II. COMPUTATIONAL MODEL
For a fully-filled guide with transverse magnetization, the nonreciprocal differential phase shift \( \Delta \phi \) is given by a relationship of the form,

\[
\Delta \phi = \Lambda \phi_o \frac{f_c}{f} \frac{\kappa}{\mu}
\]

where \( \kappa \) and \( \mu \) are effective values of the permeability tensor for the magnetized ferrite, \( f_c \) is the cutoff frequency of the waveguide, \( f \) is the operating frequency, \( \phi_o \) is the insertion phase for a TEM wave propagating through the same length in the same medium, and \( A \) is a proportionality factor accounting for relative effectiveness of the waveguide and bias field distributions. For example, \( A \) can be computed as equal to \( 4/\pi \) for a simple rectangular waveguide with equal and opposite uniform bias field level over the right and left halves of the guide. The mathematics leading to equation (1) have been derived on the basis of a transmission line equivalent-circuit model [4].

The insertion phase \( \phi_o \) can be expressed as

\[
\phi_o = \beta_o \frac{l_p}{\lambda_e} \text{ degrees}
\]

with \( l_p \) defined as the polarizer length; furthermore,

\[
f_c = \frac{c}{\lambda_c \sqrt{\mu_r \epsilon_r}}
\]

and for a circular waveguide \( \lambda_c = 1.705 \, d \), giving

\[
\Delta \phi \approx 211.1 \, A \frac{l_p}{d} \frac{\kappa}{\mu} \text{ degrees}
\]

where \( d \) is the guide diameter.

When the rod is magnetized by a weak applied field, \( \kappa \) and \( \mu \) can be approximated by [5]

\[
\kappa = \frac{M}{M_s} \frac{\omega}{\omega_m}
\]

\[
\mu = \mu_i + (1 - \mu_i) \frac{\tanh (1.25 \, M / M_s)}{\tanh (1.25)}
\]

where \( \mu_i \) is the initial permeability, given by

\[
\mu_i = \frac{1}{3} + \frac{2}{3} \left[ \left( \frac{\omega_m}{\omega} \right) \right]^2
\]
If the applied field is large enough to saturate the rod completely, then $\kappa$ and $\mu$ are more appropriately taken as

$$\kappa = - \frac{M}{M_s} \left( \frac{\omega \omega_m}{\omega_o^2 - \omega^2} \right)$$  \hspace{1cm} (8)

$$\mu = 1 - \frac{\omega \omega_m}{\omega_o^2}$$  \hspace{1cm} (9)

In the above equations $\omega$ is the operating radian frequency, $\omega_m$ is the material characteristic radian frequency equal to the product of the gyromagnetic ratio and the saturation moment, and $\omega_o$ is the resonance frequency given by Kittel’s equation,

$$\omega_o = \sqrt{\left\{ \frac{1}{2} H_o + (N_X - N_Z) \omega_m \right\} \left\{ \frac{1}{2} H_o + (N_Y - N_Z) \omega_m \right\}}$$  \hspace{1cm} (10)

Here $\gamma$ is the gyromagnetic ratio, $H_o$ is the externally applied field (z-directed), and $N_X, N_Y$ and $N_Z$ are demagnetizing factors for the ferrite shape.

The polarizer sections of concern here are usually biased near the “knee” of the magnetization curve to achieve a significant interaction with moderate mmf. This operating point does not properly conform to either the “weak-bias” or “strong bias” cases described above. However, experimental studies have yielded the following observations:

1. The coefficient $A$ is numerically equal to about 1.23.
2. The “weak-bias” model fits experimental data reasonably well for $M/M_s \leq 0.25$. Above this level, a smooth transition begins toward the “strong-bias” model.
3. The “strong-bias” model gives a good fit to experimental data for $M/M_s$ values near the “knee” of the magnetization curve.

Proceeding on this basis, take $H_o = \omega_m M/M_s$ in equation (10) and note that for a long rod transversely magnetized, with the $x$ direction along the rod axis,

$$N_X = 0; N_Y = N_Z = 1/2$$  \hspace{1cm} (11)

Then substitute into equation (10) to get

$$r = \frac{\omega_o}{\omega_m} = \left( \frac{M}{M_s} \right) \left( \frac{M}{M_s} - \frac{1}{2} \right)$$  \hspace{1cm} (12)

Now it is possible to form the ratio $\kappa/\mu$ by substituting back into equations (8) and (9), and after a little bit of manipulation to get

$$\kappa = \frac{M}{M_s} \frac{m}{\left( 1 - m^2 \frac{r}{1 + r} \right)}$$  \hspace{1cm} (13)

where $m = \omega_m/\omega$ and $r$ is defined by equation (12) above.

Then using $A = 1.23$, equation 4 for $\Delta \phi$ becomes,

$$\Delta \phi = 260.2 \frac{1}{d} \frac{M}{M_s} \frac{m}{\left( 1 - m^2 \frac{r}{1 + r} \right)}$$  \hspace{1cm} (14)

Normalized curves relating differential phase shift to the $m$ and $M/M_s$ ratios have been plotted in Figure 2 using this relationship.

III. EXPERIMENTAL VERIFICATION AND DESIGN DATA

In most commonly used ferrites, the value $M/M_s$ at the knee” of the magnetization curve is in the range 0.60 to 0.75. For example, in the case of lithium-titanium ferrite, a value of 0.75 is appropriate, so that

$$r = \sqrt{0.75 \cdot (0.75 - 0.5)} = 0.433$$  \hspace{1cm} (15)

and

$$\Delta \phi = 195.1 \frac{1}{d} \frac{m}{\left( 1 - 0.62 m^2 \right)}$$  \hspace{1cm} (16)

This expression gives results in good agreement with experimental data as plotted in Figure 3 for a C-Band circular polarizer. Similar agreement has been obtained at other frequency bands with other materials.
Solving equation (14) for \( l_p \), with \( \Delta \phi = 90^\circ \),

\[
l_p = 0.346 \frac{M_s}{M} \frac{[1 - m^2 r(l + r)]}{m}
\]  \( (17) \)

Using this expression, a family of curves of normalized polarizer length has been plotted as a function of the \( m \) and \( M_s/M \) ratios in Figure 4. Next, Figure 5 shows curves of the fractional deviation from nominal differential phase, resulting from frequency dispersion, as a function of fractional bandwidth calculated in equation (14).

## IV. Conclusion

The simple algorithm presented here provides a useful and accurate method for calculating the differential phase shift available from a circular waveguide completely filled with ferrite. It also predicts that the frequency dispersion of the differential phase shift will depend on the saturation moment of the ferrite material, becoming greater with larger moment. In addition, the model predicts that the differential phase shift will be independent of the dielectric constant of the ferrite material. Neither of these other phenomena were postulated \textit{a priori}, and yet, both have been observed experimentally on a wide variety of differential phase shift sections at different frequencies and using materials of significantly different dielectric constant.

## V. References


